SEMINAR 9

ECUAȚII

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

General form:

(1) $y^{(m)} + a_1 y^{(m-1)} + \dots + a_{m-1} y' + a_m y = f(x)$

$a_1, a_2 \dots a_m \in \mathbb{R}$ $\quad f \in C(I)$

$\longrightarrow$ nonhomogenous equation

(2) $y^{(m)} + a_1 y^{(m-1)} + \dots + a_{m-1} y' + a_m \cdot y = 0$

$\longrightarrow$ homogenous equation

Let $L[y] = y^{(m)} + a_1 y^{(m-1)} + \dots + a_m y$

$L: C^m(I) \to C(I)$ is a linear operator

We have that:

$\boxed{(1)} \iff L[y] = f$

$\boxed{(2)} \iff L[y] = 0$

The GENERAL SOLUTION of $(1) = y = \boxed{y_0} + \boxed{y_p}$, where

y_0 - the general sol. of (2)

y_p - is a particular sol of (1)

A) Case $L[y] = 0$

$\longrightarrow$ the characteristic equation:

(3) $r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0 \quad \rightarrow$ n roots

- if r is a real root of (3) with the multiplicity μ, then:

$$y_1(x) = e^{rx}$$
$$y_2(x) = x e^{rx}$$
$$\vdots$$
$$y_\mu(x) = x^{\mu-1} e^{rx}$$

- if $r = \alpha + i\beta \in \mathbb{C}$ – complex roots of (3) with multiplicity μ then:

$$y_1(x) = e^{\alpha x} \cdot \cos \beta x \qquad y_2(x) = e^{\alpha x} \cdot \sin \beta x$$
$$y_3(x) = x e^{\alpha x} \cdot \cos \beta x \qquad y_4(x) = x e^{\alpha x} \cdot \sin \beta x$$
$$\vdots \qquad \vdots$$
$$y_{2\mu-1}(x) = x^{\mu-1} e^{\alpha x} \cos \beta x \qquad y_{2\mu}(x) = x^{\mu-1} e^{\alpha x} \sin \beta x$$

$\Rightarrow \{y_1, y_2, \ldots, y_n\}$ the fundamental system of solutions

$$\boxed{y_0(x) = C_1 \cdot y_1 + C_2 y_2 + \ldots + C_n y_n, \quad C_1 \ldots C_n \in \mathbb{R}}$$

B) Case $L[y] = f$

We look for $y_p = ?$

y_p →

- can be found using the variation of constants method (S8) $\forall f(x)$; $\{y_1 \ldots, y_n\}$ f.s.s. $\Rightarrow y_p = C_1(x) y_1 + \ldots + C_n(x) y_n$

- can be found with different formulas (only in special cases for $f(x)$):

SPECIAL CASES OF $f(x)$ (we can avoid application of the var of const. method)

I. $f(x) = P_m(x)$

a) $a_m \neq 0 \Rightarrow \boxed{y_p(x) = Q_m(x)}$

b) $a_m = a_{m-1} = \dots = a_{m-p-1} = 0$ $\qquad a_{m-p} \neq 0$

$\Rightarrow \boxed{y_p(x) = x^p \cdot Q_m(x)}$

II. $f(x) = e^{rx} \cdot P_m(x)$

a) if r is not a root of (3) $\Rightarrow \boxed{y_p(x) = e^{rx} \cdot Q_m(x)}$

b) if r is a root of (3) with multiplicity $\mu \Rightarrow$

$\boxed{y_p(x) = x^{\mu} \cdot e^{rx} \cdot Q_m(x)}$

III $f(x) = e^{\alpha x} \cdot P_m(x) \cdot \cos \beta x$ / OR $f(x) = e^{\alpha x} \cdot P_m(x) \cdot \sin \beta x$

a) if $\alpha + i\beta$ is NOT a root of (3)

$\Rightarrow \boxed{y_p(x) = e^{\alpha x} \cdot \left(Q^1_m(x) \cos \beta x + Q^2_m(x) \sin \beta x \right)}$

b) if $\alpha + i\beta$ is a root of (3) with multiplicity μ

$\Rightarrow \boxed{y_p(x) = x^{\mu} \cdot e^{\alpha x} \cdot \left(Q^1_m(x) \cos \beta x + Q^2_m(x) \cdot \sin \beta x \right)}$

Ex1: $y''' - y = 0$

$A^3 - B^3 = (A-B)(A^2+B^2+AB)$
$A^3 + B^3 = (A+B)(A^2+B^2-AB)$

• we write the characteristic equation

$r^3 - r^0 = 0 \iff r^3 - 1 = 0$

$(r-1)(r^2+r+1) = 0$

$\Downarrow$ $r = 1$

$\Downarrow$ $\Delta = 1 - 4 = -3$

$\alpha = -\frac{1}{2}$ $\beta = \frac{\sqrt{3}}{2}$

$r_{2/3} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \boxed{-\frac{1}{2}} \pm i\boxed{\frac{\sqrt{3}}{2}}$

$\Rightarrow r_1 = 1 \xRightarrow{\text{real}} y_1(x) = e^{rx} = e^x$

$r_{2/3}$ complex $\Longrightarrow$ $y_2(x) = e^{\alpha x} \cos\beta x = e^{-\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2} x)$

$y_3(x) = e^{\alpha x} \cdot \sin \beta x = e^{-\frac{1}{2}x} \cdot \sin(\frac{\sqrt{3}}{2} x)$

The general solution is $y = c_1 y_1 + c_2 y_2 + c_3 y_3$

$\Rightarrow \boxed{y(x) = c_1 e^x + c_2 e^{-\frac{1}{2}x} \cos\frac{\sqrt{3}}{2}x + c_3 e^{-\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2} x)}$, $c_1, c_2, c_3 \in \mathbb{R}$

Ex2: $y''' - y'' = 0$

the characteristic equation: $r^3 - r^2 = 0$

$r^2(r-1) = 0$

$\Downarrow$ $r^1 = r_2 = 0$

$\Downarrow$ $r_3 = 1$

$r_1 = r_2 \xlongequal{\in \mathbb{R}} 0 \Rightarrow$ $\rightarrow y_1(x) = e^{r_1 x} = e^0 = 1$

$\searrow y_2(x) = x e^{r_2 x} = x e^0 = x$

$r_3 = 1 \Longrightarrow y_3(x) = e^{r_3 x} = e^x$

$\boxed{y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 + c_2 x + c_3 e^x}$, $c_1, c_2, c_3 \in \mathbb{R}$

Ex 3: $y'' + y = e^x$ → nonhomog. eq.

sol: $y = y_0 + y_p$

where — y_0 = general solution of $y'' + y = 0$
\ y_p = a particular solution of $y'' + y = e^x$

St1 $y'' + y = 0$

$r^2 + r^0 = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r_{1/2} = \pm i$ $\alpha = 0$
$\beta = 1$

$\Rightarrow y_1(x) = e^{\alpha x} \cdot \cos \beta x = \cos x$
$y_2(x) = e^{\alpha x} \cdot \sin \beta x = \sin x$

$\Rightarrow y_0 = c_1 y_1 + c_2 y_2 = c_1 \cos x + c_2 \sin x$

St2 $y_p = ?$ we have $f(x) = e^x \Rightarrow$ case B II

$f(x) = 1 \cdot e^{x \cdot 1} = P_0(x) \cdot e^{1 \cdot x}$

$r = 1$ and it's not a root of char. equation

$\Rightarrow y_p = Q_0(x) \cdot e^x = \underbrace{a}_{\text{constant}} \cdot e^x$

$y_p = a \cdot e^x$
$y'_p = a \cdot e^x \Rightarrow a \cdot e^x + a \cdot e^x = e^x$
$y''_p = a \cdot e^x$ $2a e^x = e^x \Rightarrow a = \frac{1}{2}$

$y_p = \frac{1}{2} e^x$

$$\boxed{y = y_0 + y_p \Rightarrow y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x} \quad c_1, c_2 \in \mathbb{R}$$

ex4 $y''' - y'' = x+1$ non homog. equation

$y = y_0 + y_p$

St1 $r^3 - r^2 = 0 \Rightarrow r^2(r-1) = 0$

$r_1 = r_2 = 0$ $\qquad r_3 = 1$

$r_1 = r_2 = 0$

$y_1(x) = e^{rx} = e^0 = 1$

$y_2(x) = x e^{rx} = x \cdot 1 = x$

$r_3 = 1$

$y_3(x) = e^{rx} = e^x$

$y_0 = c_1 + c_2 x + c_3 e^x$

St2

$y_p = ?$ case B, I $f(x) = P_1(x)$

y, y' don't appear $\Rightarrow$ case b where $a_n = a_{n-1} = \dots = a_{n-p-1} = 0$

$y_p = x^2 \cdot (ax+b)$

$y_p = x^2(ax+b)$

$y'_p = 2x(ax+b) + x^2 a = 3x^2 a + 2bx$

$y''_p = 6ax + 2b$

$y'''_p = 6a$

$6a - 6ax - 2b = x+1$

$-6ax + 6a - 2b = x+1$

$$\begin{cases} -6a = 1 \Rightarrow a = -\frac{1}{6} \\ 6a - 2b = 1 \Rightarrow -1 - 2b = 1 \Rightarrow -2b = 2 \Rightarrow b = -1 \end{cases}$$

$y_p = x^2(-\frac{1}{6}x - 1)$

$\boxed{y = c_1 + c_2 x + c_3 e^x + x^2(-\frac{1}{6}x - 1)}$ $c_1, c_2, c_3 \in \mathbb{R}$

Ex 5 $y'' + y = 4x \cdot e^{-x} + 2\cos x$

st 1 $r^2 + r^0 = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r_{1/2} = \pm i$

$y_1(x) = e^0 \cdot \cos x \cdot 1 = \cos x$

$y_2(x) = \sin x$

$y_0 = c_1 \cos x + c_2 \sin x$

st 2 $y_p = ?$

$$f(x) = \underbrace{4x \cdot e^{-x}}_{f_1} + \underbrace{2\cos x}_{f_2}$$

case B II case B III

THE SUPERPOSITION PRINCIPLE

For $L[y] = f_1 + f_2$, f_1, f_2 in special cases

then we look for
$$\begin{cases} y_{p1} \text{ a partic. sol of } L[y] = f_1 \\ y_{p2} \text{ a partic sol of } L[y] = f_2 \end{cases}$$

$\Rightarrow \boxed{y_p = y_{p1} + y_{p2}}$ is a PARTICULAR SOLUTION of $L[y] = f_1 + f_2$

R: $L[y_p] = L[y_{p1} + y_{p2}] = L[y_{p1}] + L[y_{p2}] = f_1 + f_2$

$f_1 = 4x \cdot e^{-x} = P_1(x) \cdot e^{-x} \Rightarrow m = 1$

$r = -1 \to$ is not a root of char. eq.

$y_{p_1} = e^{-x} \cdot (ax + b)$

$y'_{p1} = -e^{-x}(ax + b) + e^{-x} \cdot a = e^{-x}(-ax + a - b)$

$y''_{p1} = -e^{-x}(-ax + a - b) + e^{-x}(-a) = e^{-x}(ax - 2a + b)$

$$e^{-x}(ax-2a+b)+e^{-x}(ax+b)=4xe^{-x} \quad /: e^{-x}$$

$$ax-2a+b+ax+b=4x$$

$$2ax-2a+2b=4$$

$$\begin{cases} 2a=4 & \Rightarrow a=2 \\ -2a+2b=0 & \Rightarrow -4+2b=0 \Rightarrow b=2 \end{cases}$$

$$y_p=e^{-x}(2x+2)=2e^{-x}(x+1)$$

$$\longrightarrow f_2(x)=2\cos x=2\cdot e^{0x}\cdot\cos(1\cdot x)$$

$P_0=2$, $m=0$ $\quad \alpha=0 \quad \beta=1$

$\alpha+i\beta=0+i=i$ is a root of char. eq. with $p=1$

$$\Rightarrow y_{p_2}(x)=x\cdot e^{0x}\cdot(\underbrace{Q^1_0(x)}_{const}\cdot e^{0x}\cdot\cos(1\cdot x)+\underbrace{Q^2_0(x)}_{const}\cdot e^{0x}\cdot\sin x)$$

$$y_{p_2}(x)=x\cdot(a\cos(x)+b\sin x)$$

$$y'_{p_2}=a\cos x+b\sin x+x(a(-\sin x)+b\cos x)$$

$$y''_{p_2}=-a\sin x+b\cos x+a(-\sin x)+b\cos x+x(a(-\cos x)-b\sin x)=$$

$$=-2a\sin x+2b\cos x+x(-a\cos x-b\sin x)$$

$$-2a\sin x+2b\cos x+\cancel{x(-a\cos x-b\sin x)}+\cancel{x(a\cos x+b\sin x)}=2\cos x$$

$$-2a\sin x+2b\cos x=2\cos x$$

$$\begin{cases} -2a=0 & \Rightarrow a=0 \\ 2b=2 & \Rightarrow b=1 \end{cases}$$

$$\Rightarrow y_{p_2}=x(0+1\sin x)=x\sin x$$

$$y_p=y_{p_1}+y_{p_2}=x\sin x+2e^{-x}(x+1)$$

$$y=y_0+y_p$$

$$y(x)=C_1\cos x+C_2\sin x+2e^{-x}(x+1)+\sin x \qquad C_1,C_2\in\mathbb{R}$$

Ex 6 $y'' - y = \frac{2e^x}{e^x - 1}$

→ st1 $y'' - y = 0$

the char. eq: $r^2 - r^0 = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0$

$\Rightarrow r = \pm 1 \in \mathbb{R}$

$r = 1$: $y_1(x) = e^x$

$r = -1$: $y_2(x) = e^{-x}$

$\boxed{y_0 = c_1 e^x + c_2 e^{-x}}$ $c_1, c_2 \in \mathbb{R}$

st2 $y_p = ?$ $f(x) = \frac{2e^x}{e^x - 1}$

Here $f(x)$ doesn't belong to any of the special cases. We have to apply the variation of the constants method

We look for a particular sol. of the form:

$y_p(x) = c_1(x) \cdot e^x + c_2(x) \cdot e^{-x}$

y_p - sol of $y_p'' - y_p = \frac{2e^x}{e^x - 1}$

$y_p' = \underline{c_1'(x) \cdot e^x} + c_1(x) \cdot e^x + \underline{c_2'(x) \cdot e^{-x}} - c_2(x) \cdot e^x$

- we impose the condition $\boxed{c_1'(x) \cdot e^x + c_2'(x) \cdot e^{-x} = 0}$

$\Rightarrow y_p' = c_1(x) \cdot e^x - c_2(x) \cdot e^{-x}$

$y_p'' = c_1'(x) e^x + c_1(x) e^x - c_2'(x) \cdot e^{-x} + c_2(x) e^{-x}$

- we replace in equation

$c_1'(x) \cdot e^x + \underline{c_1(x) e^x} - c_2'(x) e^{-x} + \underline{c_2(x) e^{-x}} - \underline{c_1(x) e^x} - \underline{c_2(x) e^{-x}} = \frac{2e^x}{e^x - 1}$

$c_1'(x) \cdot e^x - c_2'(x) \cdot e^{-x} = \frac{2e^x}{e^x - 1}$

Thus we have the system:

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{-x} = 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} = \dfrac{2e^x}{e^x-1} \end{cases}$$

$$2c_1'(x)e^x = \frac{2e^x}{e^x-1}$$

$$c_1'(x) = \frac{1}{e^x-1} \Rightarrow c_1(x) = \int \frac{dx}{e^x-1} = \int \frac{1-e^x+e^x}{e^x-1}dx = \int\left(-1+\frac{e^x}{e^x-1}\right)dx = -x+\ln|e^x-1|$$

$$c_2'(x) = -\frac{e^{2x}}{e^x-1} \Rightarrow c_2(x) = \int -\frac{e^x.e^x}{e^x-1}dx = \int -\frac{(e^x-1+1)e^x}{e^x-1}dx =$$

$$= \int -\frac{(e^x-1)e^x}{e^x-1}dx - \int \frac{e^x}{e^x-1}dx =$$

$$= -\int e^x dx - \int \frac{e^x}{e^x-1}dx = -e^x - \ln|e^x-1|$$

$$\Rightarrow y_p(x) = (-x+\ln|e^x-1|)e^x - 1 - e^{-x}.\ln|e^x-1|$$

St3: $y = y_0 + y_p$

$$\boxed{y = c_1e^x + c_2e^{-x} + (-x+\ln|e^x-1|)e^x - 1 - e^{-x}.\ln|e^x-1|} \quad c_1, c_2 \in \mathbb{R}$$